

# Stringlike structures in the real and complex Kerr-Schild geometry

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**Abstract.** Four-dimensional (4d) Kerr-Schild (KS) geometry displays remarkable relationships with quantum world and theory of superstrings. In particular, the Kerr-Newman (KN) solution has gyromagnetic ratio  $g = 2$ , as that of the Dirac electron and represents a consistent background for gravitational and electromagnetic field of the electron. As a consequence of the very big spin/mass ratio of spinning particles, the black hole horizons disappear, obtaining the naked Kerr singular ring, which displays close kinship with the Sen fundamental string solution to heterotic string theory. The singular KN metric has to be regulated by a smooth rotating source, which sheds the light on the consistent with gravity space-time structure of the dressed electron. We consider the four-decade history of development of this structure which took finally the form of a point-string-membrane-bubble complex which is reminiscent of the enhancon model of string/M-theory. We discuss also a complex string obtained in the complex structure of the Kerr geometry. It gives an extra dimension to the world-sheet of the real Kerr string, forming a membrane by analogue with the string/M-theory unification. Finally, by analysis of the orientifold parity of the complex Kerr string, we obtain that the determined by the Kerr theorem principal null congruence of the Kerr geometry is described by a quartic equation in the projective twistor space  $CP^3$ , and therefore, it creates the known Calabi-Yau twofold (K3 surface) in twistor space of the 4d KS geometry. This strong parallelism of the 4d KS geometry with basic structures of string/M-theory confirms important role of the KS geometry for quantum physics and allows us to assume that complexification of 4d KS geometry may represent an alternative to compactification of higher dimensions. We connect this parallelism with mysterious  $N=2$  superstring which has (complex) critical dimension two and may be embedded into complex KS geometry.

## 1. Introduction

It is now generally accepted that black holes are to be associated with elementary particles. New ideas and methods in the physics of black holes (BH) are based on the complex analyticity and conformal field theory, that unites the physics of black holes with superstring theory and particle physics. In spite of these evident relationships, the path from superstring theory to particle physics represents an unsolved problem, and moreover, it is constated that "... realistic model of elementary particles still appears to be a distant dream...", John Schwarz [1]. Meanwhile, the four-dimensional Kerr-Newman solution for a charged and rotating BH [2] demonstrates in this relation some interesting surprises. Namely, it exhibits wonderful connections with spinning particles and simultaneously a remarkable parallelism with basic structures of superstring theory.

In 1968 Carter noticed that the KN solution has gyromagnetic ratio  $g = 2$ , as that of the Dirac electron [3], and therefore, at least the asymptotic electromagnetic (em) and gravitational fields

of the electron should correspond to the KN solution with great precision. Angular momentum of the KN solution  $J = ma$  is proportional to mass  $m$  and to Kerr's parameter  $a$ , which is radius of the Kerr singular ring. The em and gravitational fields are concentrated near the Kerr ring forming a type of waveguide, which causes its similarity with a closed string of the dual models. It has been assumed in [4, 5] that the em and gravitational traveling waves can propagate along the Kerr ring in analogue with excitations of the dual string models. Masses of elementary particles are very small with respect to their spin, and the Kerr parameter of rotation  $a = J/m$ , radius of the Kerr singular ring, turns out to be of order of the Compton wave length of the electron  $\sim \hbar/(2mc)$ . In the dimensionless units  $c = G = \hbar = 1$ , the ratio  $a/m$  is about  $10^{44}$ , which results in disappearance of the BH horizons, and the Kerr singular ring turns out to be naked. Other spinning particles have approximately the same ratio  $a/m$ , and we arrive to trivial, but very important fact that *black holes cannot apparently be associated with spinning particles*. The relevant to spinning particles geometry should be an over-rotating KN background which contains a closed string-like singularity.

It looks very bad, since instead of the consistent with quantum theory very weak gravitational field in vicinity of the electron, we reveal a singular core in the form of a closed string, which represents a very strong defect of the background. One more trouble of this result is a twosheeted structure of the resulting Kerr background, since the naked Kerr singular ring represents a branch line of the Kerr space-time into two sheets. This mysterious structure of the over-rotating KN geometry created the problems in modelling of the source of the Kerr and KN solutions, which could consistently regulate this background. Structure of such a source was specified step by step in many works during more than four decades, in particular in the following papers [6, 7, 5, 4, 8, 9, 10, 11, 12, 13]. As a result, the consistent regular source of the external KN solution, generating the necessary very weak gravitational field in the vicinity of the source, acquired the form of a relativistically rotating string-membrane-bubble complex: a highly oblate bubble bounded by an ellipsoidal membrane and by a circular fundamental string positioned on the edge rim of the bubble. The circular string appears as a result of regularization of the Kerr singular ring, and we argue that traveling waves circulating along this string should create a singular point, D0-brane which exhibits "zitterbewegung" of the Dirac electron. It was also shown [13] that wave excitations of the fundamental Kerr string generate also the axial half-strings which are going to infinity serving as carriers of de Broglie waves.

Along with circular fundamental string [12, 4, 14], an open complex string was obtained in the complex structure of the Kerr-Schild geometry [15, 16].<sup>1</sup> It should also be mentioned that the open Kerr string extends the world-sheet of the closed Kerr string to two-brane source, displaying parallelism with the process of string/M-theory unification [18], and moreover, the resulting source of the KN solution is reminiscent in many details the enhancon model used by the superstring/M-theory unification [19]. Finally, it has been obtained recently [20], that the twistorial structure of the four-dimensional KS geometry, being combined with the orientifold structure of the complex Kerr string, creates a K3 surface (the Calabi-Yau twofold) on the projective twistor space, which forms a holographic skeleton of the KS space, determined by the Kerr theorem. It confirms close relationships of the KS geometry with the basic structures of the superstring theory and hints existence of some underlying theory which may unify the Kerr's gravity with physics of elementary particles and superstring theory. Twistor theory plays apparently principal role in this unification, since the complex N=2 critical superstring [21] has also the inherent twistorial structure [22, 23], which allows it to be consistently embedded in the *complex* four-dimensional KS geometry. We start paper by treatment of the real, complex, twistorial and stringy structures of KS geometry along our previous papers [15, 16, 12, 20, 24].

<sup>1</sup> Both of these strings were rediscovered recently by Adamo and Newman [17] by study of the asymptotic form of the geodesic and shear-free congruences. They comment this discovery writing that "...It would have been a cruel god to have layed down such a pretty scheme and not have it mean something deep."

## 2. Real structure of the Kerr-Newman (KN) geometry

The KN metric is represented in the Kerr-Schild (KS) form [2],

$$g_{\mu\nu} = \eta_{\mu\nu} + 2he_\mu^3 e_\nu^3, \quad (1)$$

where  $\eta_{\mu\nu}$  is auxiliary Minkowski background in Cartesian coordinates  $x = x^\mu = (t, x, y, z)$ ,

$$h = P^2 \frac{mr - e^2/2}{r^2 + a^2 \cos^2 \theta}, \quad P = (1 + Y\bar{Y})/\sqrt{2}, \quad (2)$$

and  $e^3(x)$  is a tangent direction to a *Principal Null Congruence (PNC)*, which is determined by the form<sup>2</sup>

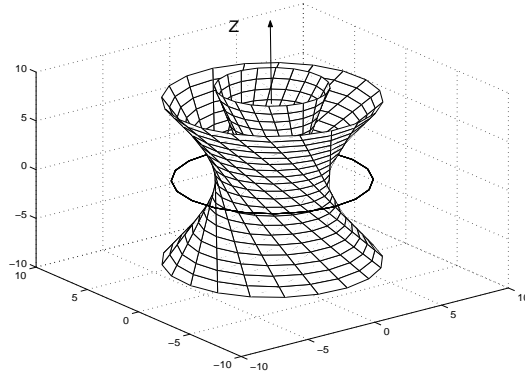
$$e_\mu^3 dx^\mu = du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y\bar{Y} dv, \quad (3)$$

via function  $Y(x)$ , which is controlled by the *Kerr theorem*, [2, 36, 25, 26].

The twisting lightlike rays of the Kerr PNC are focussing in the equatorial plane  $\cos \theta = 0$ , at the Kerr singular ring,  $r = 0$ , approaching it tangentially. As a result, the aligned with Kerr PNC metric and the KN electromagnetic potential,

$$A_\mu = -P^{-2} \text{Re} \frac{e}{(r + ia \cos \theta)} e_\mu^3, \quad (4)$$

concentrate near the Kerr ring and form a closed lightlike gravitational waveguide [4], playing the role of a closed string which may carry excitations in the form of the lightlike traveling waves [5, 13], which take near the ring the form of pp-waves considered as string excitations in low energy string theory.



**Figure 1.** Twistor null lines of the Kerr congruence are focused on the Kerr singular ring which forms a branch line of space in two sheets.

### 2.1. Regular source of the KN solution

Gyromagnetic ratio of the KN solution is  $g = 2$ , as that of the Dirac electron [3, 2]. Therefore, the observable parameters of the electron: mass, spin, charge and magnetic moment indicate that the KN gravitational field corresponds to the electron background geometry [13], which created a series of works on the consistent with gravity classical model of the electron

<sup>2</sup> Here  $\zeta = (x + iy)/\sqrt{2}$ ,  $\bar{\zeta} = (x - iy)/\sqrt{2}$ ,  $u = (z + t)/\sqrt{2}$ ,  $v = (z - t)/\sqrt{2}$ , are the null Cartesian coordinates,  $r, \theta, \phi$  are the Kerr oblate spheroidal coordinates, and  $Y(x) = e^{i\phi} \tan \frac{\theta}{2}$  is a projective angular coordinate. The used signature is  $(-+++)$ .

[7, 9, 5, 13, 27, 28, 29, 30] and stimulated investigations of models of the source of the Kerr and KN solution.

(1) *The Kerr ring as a closed string.* This model, created first in 1975 [4], by analogue with the Dirac string of magnetic monopole, was supported then by the studies of the fundamental string solutions to low-energy string theory [31, 32]. In 1992 Sen obtained two important solutions to low-energy string theory: a) solution for fundamental heterotic string [33], and b) analogue of the Kerr solution to low-energy string theory [34]. It has been shown then in [14] that metric and electromagnetic field near the Kerr ring in the solution (b) have the lightlike Killing structure similar to that of the fundamental heterotic string solution (a). The only difference consisted in branching of the Kerr singularity related with two sheets of the Kerr geometry.

(2) *String-membrane source.* Alternative line of investigations, related with attempts to remove Kerr's twosheetedness was started by H. Keres [6], who noticed that the complex radial distance  $\tilde{r} = r + ia \cos \theta$ , which determines the KN gravitational and electromagnetic potential  $\Phi = \text{Re}(m/\tilde{r})$ , being expressed in the Cartesian coordinates yields  $(\tilde{r})^2 = x^2 + y^2 + (z - ia)^2$ , and therefore, "function  $\tilde{r}^{-1}$  and its first derivatives are continuous over all space, beside the disk  $z = 0, x^2 + y^2 \leq a^2$ ," and "... crossing the disk the derivatives have a jump discontinuity..." indicating that the sources should lie on the disk. Similarly, Israel truncated the Kerr 'negative' Kerr sheet along the disk  $r = 0$ , and showed in [7] that the resulting jump discontinuity should create a matter distribution over the disk surface. He interpreted the disk-like source of the Kerr geometry as a classical model of electron. Next important step was done by Hamity [8], who showed that Kerr's disk has to be rigidly rotating, reaching the velocity of light on the string-like border of the disk. Finally, López [9] constructed the regular source of the KN solution, excising the singular region  $r < r_e = e^2/2m$  (note that  $r$  is the Kerr ellipsoidal radial coordinate), and the Kerr-Newman (KN) regular source took the form of a rigidly rotating ellipsoidal membrane, or the disk-like bubble, flat interior of which is matched with the external KN solution along the bubble surface  $r = r_e$ , being determined unambiguously by the equation  $h(r) = 0$ , where  $h$  is given by (2).

As a result, the gravitational singularity disappears, and the space around the regularized disk-like source (= Compton region of the dressed electron) turned out to be very close to flat. In the same time, the singular electromagnetic field gets cutoff at  $r = r_{reg} = r_e = e^2/2m$  (ellipsoidal radial coordinate) and is regularized. It turns the Kerr ring into regular fundamental string positioned at the edge rim of the ellipsoidal bubble. As it was shown in [13], the traveling waves along circular string create the axial singular beams (outgoing pp-wave half-string) which serve as carriers of de Broglie waves.

(3) *Gravitating soliton model.* In [12], the membrane-bubble model was extended to a smooth field model based on the system of chiral superfields and on the suggested in [10] modified form of the KN metric. The regular bubble-source was formed by a domain wall phase transition interpolating between the external KN background and a supersymmetric pseudovacuum state inside the bubble. The model represents a soliton of the oscillon type, since the internal false-vacuum state is built of the oscillating Higgs-like field. The KN vector potential is dragged by the relativistically rotating string and forms a closed Wilson loop, which turns out to be quantized via interaction with the internal Higgs field, resulting in a consistent with KN gravity rotating soliton (for details see [12]).

The KN electromagnetic field is regularized and forms a lightlike fundamental string positioned on the boundary of the disk-like bubble in equatorial plane  $\theta = \pi/2$ . New important effect is related with circular traveling waves which should propagate along fundamental string distorting the form of the bubble surface. The cutoff  $r_e$  is determined by amplitude of the vector-potential, and there appear some nodes where the cutoff  $r_e$  approaches zero, and regularization is to be broken. The deformed by waves bubble boundary touches the Kerr singular ring creating singular points, D0-branes which circulate along the bubble in equatorial plane. These singular

points form the ends of the fundamental string and can be identified with lightlike quarks or partons. It turns out to be closely related with the fact that the fundamental circular string, being parametrized by the light cone parameter  $\sigma = t - n\phi$ , cannot indeed be closed, since the end points  $\phi = 0$  and  $\phi = \pi$  turn out to be disconnected by a time-like gap  $x^\mu(t-0) - x^\mu(t-n\pi)$ , which should be closed by the path of a massive mode. These end points are stuck at the bubble, which therefore acquires the role of a heavy quark which closes this loop. It suggests a physical mechanism which can lie beyond “zitterbewegung” of the Dirac electron. On the other hand, it resolves the seeming contradiction between the point-like and extended electron: the naked electron is point-like and represents a circulating singular pole (D0-brane), while the dressed KN electron exhibits an unambiguously defined extended structure of the bag model, which is similar to the MIT and SLAC bag models, but is consistent with KN gravity. From geometrical point of view, it represents a simplicial complex of the Compton size, consisting of the point, circular string, ellipsoidal membrane and a false-vacuum bubble filled by Higgs field – the complex system of the D0-D1-D2-D3 branes.

(4) *Complex Kerr string.* Along with the circular string structure, the Kerr-Schild geometry contains also a *complex string* [15, 16], which appears in the initiated by Newman complex representation of the Kerr geometry [35]. This string is open and parametrized by angular variable  $\theta \in [0, \pi]$ . In fact, parameter  $\theta$  extends the world-sheet of the circular Kerr string (positioned at  $\theta = \pi/2$ ) to the membrane source of the López bubble model [9, 12]. Structure of the regular bubble source of the KN solution represents a copy of the enhancon model used for regularization by the superstring/M-theory unification [19] and represents a counterpart of the transfer from 10-dimensional superstrings to 11d  $M$ -theory [18].

## 2.2. The Kerr theorem and twistorial structure of the KS geometry

**The Kerr Theorem** [2, 24, 36] determines the shear-free null congruences with tangent direction (3) by means of the solution  $Y(x)$  of the equation

$$F(T^A) = 0, \quad (5)$$

where  $F(T^A)$  is an arbitrary holomorphic function in the projective twistor space

$$T^A = \{Y, \quad \lambda^1 = \zeta - Yv, \quad \lambda^2 = u + Y\bar{\zeta}\}. \quad (6)$$

Using the Cartesian coordinates  $x^\mu$ , one can rearrange variables and reduce function  $F(T^A)$  to the form  $F(Y, x^\mu)$ , which allows one to get solution of the equation (5) in the form  $Y(x^\mu)$ .

For the Kerr and KN solutions, the function  $F(Y, x^\mu)$  turns out to be quadratic in  $Y$ ,

$$F(Y, x^\mu) = A(x^\mu)Y^2 + B(x^\mu)Y + C(x^\mu), \quad (7)$$

and the equation (5) represents a *quadric* in the projective twistor space  $CP^3$ , with a non-degenerate determinant  $\Delta = (B^2 - 4AC)^{1/2}$  which defines the complex radial distance [25, 26, 24]

$$\tilde{r} = -\Delta = -(B^2 - 4AC)^{1/2}. \quad (8)$$

This case is explicitly resolved and yields two solutions

$$Y^\pm(x^\mu) = (-B \mp \tilde{r})/2A, \quad (9)$$

which allows one to restore two PNC by means of (3). One can easily obtain from (7) and (9) that the used in the metric (2) and the em potential (4) complex radial distance  $\tilde{r} = r + ia \cos \theta$  may also be determined from the Kerr generating function by the relation

$$\tilde{r} = -dF/dY. \quad (10)$$

Therefore, the Kerr singular ring,  $\tilde{r} = 0$ , is formed as a caustic of the Kerr congruence,

$$dF/dY = 0. \quad (11)$$

As a consequence of Vieta's formulas, the quadratic in  $Y$  function (7) may be expressed via the solutions  $Y^\pm(x^\mu)$  in the form

$$F(Y, x^\mu) = A(Y - Y^+(x^\mu))(Y - Y^-(x^\mu)). \quad (12)$$

It is known, [2], that the linear in  $Y$  generating functions  $F$  correspond to the light-like solutions, while the subsequent analysis of the generating functions of higher degrees in  $Y$  showed that they split into products of the blocks of the first and second degrees in  $Y$ , and therefore, they should correspond to multi-particle KS solutions [37, 38, 39].

### 3. Complex Kerr geometry and an open complex string

One can see that the complex radial distance  $\tilde{r} = r + ia \cos \theta$  takes in the Cartesian coordinates the form

$$\tilde{r} = \sqrt{x^2 + y^2 + (z + ia)^2}, \quad (13)$$

and therefore, the scalar component of the vector potential (4) may be obtained from the Coulomb solution  $\phi(\vec{x}) = e/r = e/\sqrt{x^2 + y^2 + z^2}$  by a complex shift  $z \rightarrow z + ia$ , or by the shift of its singular point  $\vec{x}_0 = (0, 0, 0)$  in complex region  $\vec{x}_0 \rightarrow (0, 0, -ia)$ . The complex shift was first considered by Appel in 1887 [40], who noticed that the Coulomb solution, being invariant solution to the linear Laplace equation with respect to real shifts of its origin  $\vec{x} \rightarrow \vec{x} + \vec{a}$ , should also be invariant with respect to the complex shift. In spite of triviality of this procedure from complex point of view, it yields very nontrivial consequences in the real section, in particular, the singular point of the Coulomb solution  $\vec{x}_0 = (0, 0, 0)$  turns into singular ring  $x^2 + y^2 + (z + ia)^2 = 0$  (intersection of the sphere  $x^2 + y^2 + z^2 = a^2$  and the plane  $z = 0$ ), which becomes the branch line of the space into two sheets.

Newman showed that the KN electromagnetic field and the linearized KN gravitational field may be obtained from a complex retarded-time construction as the fields generated by a complex source propagating along a complex world line (CWL) [35]. Later on, it has been shown that linearized Newman's representation acquires exact meaning in the Kerr-Schild class of metrics, [5, 24, 25, 26], and the KN solution corresponds exactly to the field generated by a *complex source propagating along straight CWL*

$$x_L^\mu(\tau_L) = x_0^\mu(0) + u^\mu \tau_L + \frac{ia}{2} \{k_L^\mu - k_R^\mu\}, \quad (14)$$

where  $u^\mu = (1, 0, 0, 0)$ ,  $k_R = (1, 0, 0, -1)$ ,  $k_L = (1, 0, 0, 1)$  and  $\tau_L = t_L + \sigma_L$  is a complex retarded-time parameter. Index  $L$  labels it as a Left structure, and we should also add a complex conjugate Right structure

$$x_R^\mu(\tau_R) = x_0^\mu(0) + u^\mu \tau_R - \frac{ia}{2} \{k_L^\mu - k_R^\mu\}. \quad (15)$$

Therefore, from complex point of view the Kerr and Schwarzschild geometries are equivalent and differ only by the relative position of their *real slice*, which for the Kerr solution goes aside of its center. Complex shift turns the Schwarzschild radial directions  $\vec{n} = \vec{r}/|r|$  into twisted directions of the Kerr congruence, see Fig. 1.

Principal part of the complex retarded-time construction is a family of the complex light cones  $\mathcal{K}$  adjoined to CWL. Taking the Left CWL as a basis for treatment, one can represent the family of the adjoined complex light cones in spinor form

$$\mathcal{K}_L = \{x : x = x_L^i(\tau_L) + \psi_L^A \sigma_{AA}^i \tilde{\psi}_R^{\dot{A}}\}. \quad (KL)$$

The cones are split into two families of null planes: "left" ( $\psi_L = \text{const.}; \tilde{\psi}_R - \text{var.}$ ) and "right" ( $\tilde{\psi}_R = \text{const.}; \psi_L - \text{var.}$ ). These are the only two-dimensional planes which are wholly contained in the complex null cone. The rays of the principal null congruence of the Kerr geometry are the tracks of these complex null planes (right or left) on the real slice of the Minkowski background. The real null direction  $e^3$ , given by (3), is a projective version of the spinor form  $\psi_L^A \sigma_{AA}^i \tilde{\psi}_R^{\dot{A}}$ , expressed via the first projective twistor coordinate  $Y = \psi_L^1 / \psi_L^0$ . The real null direction  $e^3$  is completed by two complex conjugate null forms

$$e^1 = d\zeta - Y dv, \quad e^2 = d\bar{\zeta} - \bar{Y} dv. \quad (16)$$

One sees that the second projective twistor coordinate  $\lambda^1 = \zeta - Yv = x^\mu e_\mu^1$  represents projection of the space-time point  $x^\mu$  on the null direction  $e^1$ , while the third projective twistor coordinate  $\lambda^2 = u + Y\bar{\zeta} = x^\mu (e_\mu^3 - \bar{Y}e_\mu^1)$  represents a linear combination of the projections on the null directions  $e^3$  and  $e^1$ . The determined by the Kerr theorem function  $Y(x)$  allows one to restore at each point  $x$  the remaining twistor coordinates  $\lambda^1$  and  $\lambda^2$ , and to fix the incident Left null plane spanned by the null directions  $e^3$  and  $e^1$ . As a consequence, the solution of the Kerr theorem  $Y(x)$  foliates the Minkowski space-time into the Left null planes, which due to specific structure of the KS metric (1) turn out to be null with respect to the curved KS space-time too, and therefore, the Kerr theorem performs a specific linearization of the curved KS space-times, which justifies validity of the twistor version of the Fourier transform [41].

Similarly, the Right null planes are to be spanned by the null directions  $e^3$  and  $e^2$ . The twistor null rays of the Kerr congruence are formed as intersections of the complex conjugate Left and Right null planes.

### 3.1. Complex open string

It was obtained in [15, 16, 22] that the complex world line  $x_0^\mu(\tau)$ , parametrized by complex time  $\tau = t + i\sigma$ , represents really a two-dimensional surface which takes an intermediate position between world-line of the particle and the string world-sheet. The corresponding "hyperbolic string" equation [22],  $\partial_\tau \partial_{\bar{\tau}} x_0(t, \sigma) = 0$ , has the general solution

$$x_0(t, \sigma) = x_L(\tau) + x_R(\bar{\tau}) \quad (17)$$

as sum of the analytic and anti-analytic modes  $x_L(\tau)$ ,  $x_R(\bar{\tau})$ , which are not necessarily complex conjugate. The complex light cones  $\mathcal{K}$  adjoined to each point of the world-sheet  $x_0^\mu(\tau)$  are split into the Left and Right null planes, forming the linear N=2 structure of the complexified KS geometry. The real KS structure appears as a result of *projection of this structure on the real slice, which is non-linear operation*, resulting in twisted structure of the real KS space-time.

For each real point  $x^\mu$ , the parameters  $\tau$  and  $\bar{\tau}$  are to be determined by a complex retarded-time construction. Complex source of the KN solution corresponds to two *straight* complex conjugate world-lines (14), (15). Contrary to the real case, the complex retarded-advanced times  $\tau^\mp = t \mp \tilde{r}$  may be determined by two different (Left or Right) complex null planes, which are generators of the complex light cone. It yields four different roots for the Left and Right complex structures [25, 26, 24]

$$\tau_L^\mp = t \mp (r_L + ia \cos \theta_L), \quad (18)$$

$$\tau_R^\mp = t \mp (r_R + ia \cos \theta_R). \quad (19)$$

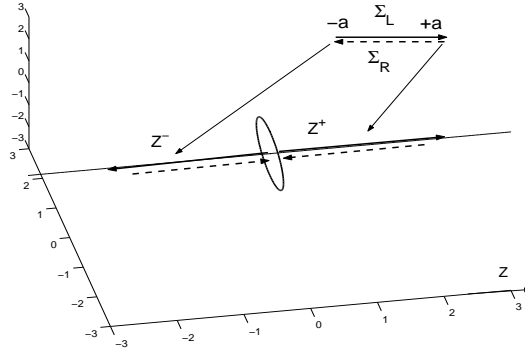
The real slice condition determines the relation  $\sigma = a \cos \theta$ , which connects parameter  $\sigma$  with angular directions of the Kerr congruence  $\theta \in [0, \pi]$ . It puts restriction  $\sigma \in [-a, a]$  indicating that *the complex string is open*, and its endpoints  $\sigma = \pm a$  may be associated with the Chan-Paton charges of a quark-antiquark pair. In the real slice, the complex endpoints of the string are mapped to the north and south twistor null lines,  $\theta = 0, \pi$ , see Fig. 3.

### 3.2. Orientifold projection

Boundary conditions of the complex open string cannot consistently be set for its real and complex part simultaneously [15, 16]. Solution of this obstacle requires an *orientifold* structure [18, 21] of the worldsheet of the open complex string. Orientifold turns the open string in a closed but folded one. The world-sheet parity transformation  $\Omega : \sigma \rightarrow -\sigma$  reverses orientation of the world sheet, and covers it second time in mirror direction. Simultaneously, the Left and Right modes are exchanged.<sup>3</sup> The projection  $\Omega$  is combined with space reflection  $R : r \rightarrow -r$ , resulting in  $R\Omega : \tilde{r} \rightarrow -\tilde{r}$ , which relates the retarded and advanced folds

$$R\Omega : \tau^+ \rightarrow \tau^-, \quad (20)$$

preserving analyticity of the world-sheet. The string modes  $x_L(\tau)$ ,  $x_R(\bar{\tau})$  are extended on the



**Figure 2.** Ends of the open complex string, associated with quantum numbers of quark-antiquark pair, are mapped onto the real half-infinite  $z^+$ ,  $z^-$  axial strings. Dotted lines indicate orientifold projection.

second half-cycle by the well-known extrapolation [18, 21]

$$x_L(\tau^+) = x_R(\tau^-); \quad x_R(\tau^+) = x_L(\tau^-), \quad (21)$$

which forms the folded string, in which the retarded and advanced modes are exchanged every half-cycle.

The presented in Fig. 2 diagram shows a crossing symmetry of the four roots  $\tau^\pm$  and  $\bar{\tau}^\pm$ , for the complex retarded time, which allows one to replace the Right complex conjugate retarded-time structure  $x_R(\tau^-)$  by the antipodal Left advanced-time structure  $x_L(\tau^+)$ , and works only in terms of the Left complex structures, omitting the index ‘L’. For more details on this antipodal relation see [42, ?].

## 4. Calabi-Yau twofold from the Kerr theorem

### 4.1. Problem of the non-stationarity

The algebraically special exact KS solutions of the Einstein-Maxwell equations obtained in the remarkable paper [2] have two essential restrictions. First one is requirement of stationarity. In particular, the Kerr and KN solutions are stationary and their nonstationary generalizations are unknown. In the KS formalism [2] stationarity is characterized by a constant Killing direction  $K^\mu$ , which corresponds to invariance of the metric  $g_{\mu\nu}$  with respect to action of the operator

<sup>3</sup> Two oriented copies of the interval  $\Sigma = [-a, a]$ ,  $\Sigma^+ = [-a, a]$  and  $\Sigma^- = [-a, a]$ , are joined, forming a circle  $S^1 = \Sigma^+ \cup \Sigma^-$ , parametrized by  $\theta$ , and map  $\theta \rightarrow \sigma = a \cos \theta$  covers the world-sheet twice.



$\hat{K} = K^\mu \partial_\mu$ , i.e.  $\hat{K}g^{\mu\nu} = 0$ . Using Cartesian coordinates, one can express  $K^\mu$  via coordinates of the straight CWL  $x_0^\mu(\tau)$ , complex source of the Kerr and KN solutions, and obtain

$$K^\mu = \partial_\tau x_0^\mu(\tau). \quad (22)$$

Stationarity of the metric implies stationarity of the Kerr congruence  $\hat{K}e^3 = 0$ , which imposes the corresponding restriction on the coefficients  $A, B, C$  of the generating function of the Kerr theorem and the corresponding solutions

$$\hat{K}Y = \hat{K}\bar{Y} = 0. \quad (23)$$

All these functions turn out to be functions of the coordinates  $x_L^\mu(\tau^-)$  and 4-velocity of the CWL,  $u^\mu(\tau^-) = \dot{x}_L^\mu(\tau^-) \equiv \partial_\tau x_L^\mu(\tau)|_{\tau^-}$ , and finally they should depend on the complex retarded time parameter  $\tau^-$ .<sup>4</sup> Although the determined by the Kerr theorem roots  $Y^\pm(x, \tau^-)$  depend formally on the retarded time  $\tau^-$ , *for the straight CWL* of the Kerr and KN solutions this dependence drops out from the final expressions, reducing to the dependence of the functions  $F$  and  $Y$  only on 4-velocity  $u^\mu(\tau^-)$ , or on the related parameters  $K^\mu(\tau)$  [24].

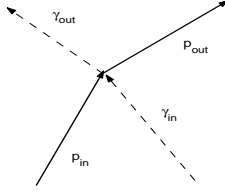
The second restriction is related with idealization of the situation.

The non-stationary (accelerating) KS solutions are unknown. However, the known Kinnersley solutions for accelerating non-rotating sources demonstrate that acceleration is always accompanied by radiation. The obtained in [43, 44] exact non-stationary em solutions on the stationary KS background show also that any em excitation of the Kerr geometry creates specific radiation in the form of singular beams (pp-waves) propagating along the null lines of the Kerr congruence. These beams are incoming on the negative sheet of the Kerr geometry, and propagate towards the disk  $r = 0$ , when passing analytically through the disk, the beam-pulses become outgoing and propagate towards future infinity. Some of the exact non-stationary em solutions (pp-waves) do not create additional radiation and do not cause the recoil. However, these exact stationary solutions correspond to an idealized situation with a congruence, which transfers analytically from the negative to positive sheets of KS space-time and corresponds to an isolated stationary system without internal and external sources.

When we consider the regularized KN solution, the idealized analytic KN background is to be replaced by the background with a membrane source separating the bulk of the positive Kerr sheet,  $r \geq r_e$ , from the bulk of the negative sheet,  $r \leq r_e$ . The membrane source breaks analyticity of the Kerr solutions, and results in the scattering of the pp-wave radiation creating recoil and non-stationarity of the KS solutions. In the moment of the collision of the beam pulse with the membrane-source which closes the gate from negative to positive sheet at  $r = r_e$ , there should appear a kink in the parameters of CWL. Parameters of the Kerr ingoing congruence  $A_{in}, B_{in}, C_{in}$  should differ from parameters of the outgoing one,  $A_{out}, B_{out}, C_{out}$ . The in- and out-sources of the Kerr geometry turn out to be decorrelated and should be considered as independent sources, generating different congruences. In the simplest case, we can set  $r_e = 0$ , which corresponds to Keres-Israel-Hamity disklike membrane source which blocks the gate from the negative to positive sheet, the ingoing (real or virtual) photon will be scattered by the membrane and produce a recoil, causing a jump,  $K_{in}^\mu \neq K_{out}^\mu$ , described by a typical Feynman graph.

In the modern physics, treatment of the classical problem of radiation reaction has reemerged in the connection with gravitational radiation and problems of quantum gravity. In the seminal work by DeWitt and Brems [45], the so-called ‘bi-tensor’ fields and Hadamard’s ‘elementary solution’ were considered as a useful instrument for study of this problem in analogue with the two-point Green functions and the Feynman propagator. One can note that orientifold represents a stringy analog of the ‘bi-tensor’ fields.

<sup>4</sup> Details of these relations are not essential for our treatment here and can be found in [24, 25, 26].



**Figure 3.** Recoil of the CWL caused by an electromagnetic beam or by a photon propagating from the past infinity through the Kerr ring to the future infinity.

Due to kink of the trajectory at the moment  $\tau_0$ , the relationship between the retarded and advanced roots is to be biased (as it is shown in the diagram of Fig. 2), and the retarded coefficients of the Kerr generating function  $A(\tau^-)$ ,  $B(\tau^-)$ ,  $C(\tau^-)$  should not match in general with the coefficients  $A(\tau^+)$ ,  $B(\tau^+)$ ,  $C(\tau^+)$ , corresponding to advanced time  $\tau^+$ . As a result, the retarded and advanced world lines,  $x_L^\mu(\tau^-)$  and  $x_L^\mu(\tau^+)$ , generating two different functions,  $F^+$  and  $F^-$ , and correspondingly, two different twistorial manifolds have to be determined by a two-particle version of the Kerr theorem.

In accordance with the treatment of the multi-particle Kerr-Schild solutions [37, 38, 39], generating function of the two-particle system has to be determined by the function  $F^{(2)}(T^A) = F^+(T^A)F^-(T^A)$ , composed as a product of two functions  $F^+$  and  $F^-$  corresponding to the retarded,  $x_L(\tau^-)$ , and advanced,  $x_L(\tau^+)$ , complex sources. The function

$$F^{(2)}(T^A, x_L^+, x_L^-) = F^+(T^A, x_L^+)F^-(T^A, x_L^-) \quad (24)$$

represents a bi-scalar, or two-point function associated with the world-sheet parity,  $\tau_L^- \leftrightarrow \tau_L^+$ .

Both factors,  $F^+$  and  $F^-$  are quadratic in  $T^A$ , and each of the partial equations  $F^+ = 0$  (or  $F^- = 0$ ) generates a *quadric* in the projective twistor space  $CP^3$  corresponding to the usual two-sheeted structure of the Kerr geometry with two different real Killing directions consistent with the Debney-Kerr-Schild formalism.

The ‘product’ manifold, determined by the equation  $F^{(2)}(T^A, x_L^+, x_L^-) = 0$ , corresponds to a fourfold described as a *quartic* in the projective twistor space  $CP^3$ , which is the Calabi-Yau (complex) twofold, or the well-known K3 surface used in diverse models of the string compactification and also by generalization of superstring theory to M-theory [18, 21]. We obtain here that dynamical generalization of the Kerr geometry requires splitting of the complex source of the Kerr geometry into independent retarded and advanced components described by the orientifold parity of the world-sheet, and application of the Kerr theorem creates the K3 surface in the projective twistor space  $CP^3$ .

#### 4.2. Matching of the retarded and advanced fields

One sees that the fields at a fixed *real* point  $x^\mu$  near the KN source may be represented as a sum of the retarded field generated by the past complex light cone (root  $\tau^-$ ) and the advanced field generated by the future complex light cone (root  $\tau^+$ ). For the stationary case, these points lie on the same straight CWL and generate two congruences of the same 3d form, which differ by their space-orientation. The generated by  $\tau^-$  congruence is *outgoing* on the physical (positive) sheet,  $r > 0$ , while the congruence generated by  $\tau^+$  is formed by the *ingoing* rays, and therefore, its physical sheet is ‘negative’, and corresponds to  $r < 0$  for the root  $\tau^+$ . So far as the momenta of the complex Kerr source  $u_{in}^\mu$  and  $u_{out}^\mu$  are to be real, there may be set the time-ordering  $\tau^- \prec \tau^+$ , which shows that the retarded physical sheet precedes the sheet of advanced fields.

The four sheets of these congruences may be marked by the four solutions  $Y^\pm(x, \tau^\pm)$  and  $Y^\pm(x, \tau^-)$ . For the stationary KN solution, the Kerr generating function  $F(Y, x(\tau))$  is

independent of  $\tau$ , and as a result  $F^+(Y) = F^-(Y)$ , and the four congruences coincide pairwise,  $Y^\pm(x, \tau^+) = Y^\mp(x, \tau^-)$ , forming a *global* orientifold parity for the stationary Kerr geometry.

The orientifold condition (21) matches the lines of Kerr congruences at the world-sheet of the complex Kerr string. On the real slice of the Kerr geometry, it induces matching of the Kerr congruence at the membrane-surface of the regularized KN solution which separates the external KN solution from the internal false-vacuum state. As we discussed in Sec. 2, in the simplest Keres-Israel-Hamity model of the KN source, this membrane represents the disk  $r = 0$ , which separates the ‘positive’ and ‘negative’ sheets of the Kerr geometry. The closed lightlike Kerr string lies at the boundary of the disk at  $r = 0$  and  $\cos \theta = 0$ , and this string is extended to membrane by the extra world-sheet parameter  $\theta \in [0, \pi]$ . One sees here close parallelism with the superstring/M-theory correspondence, where the heterotic string transfers to a membrane growing by the eleventh dimension [18].

The source of the *gravitating soliton* model represents a false-vacuum bubble, boundary of which represents a domain wall membrane providing smooth phase transition from external KN solution to a flat regular internal space. The membrane-boundary of the bubble has ellipsoidal form determined by the surface  $r = r_{reg}$ , and therefore, it fixes the boundary of analyticity of the KN geometry, or the boundaries of the ‘physical sheets’ for the advanced and retarded fields

$$\tau_L^-|_{r=r_{reg}} = t - r_{reg} - ia \cos \theta \equiv t - (r_{reg} + ia \cos \theta) = (\tau_R^-)|_{r=r_{reg}}^* = \tau_L^+|_{r=r_{reg}}. \quad (25)$$

It means that analyticity of the Kerr congruence is to be broken on the boundary  $r = r_{reg}$ , where the retarded ‘outgoing’ Kerr congruence should be replaced by the ‘ingoing’ advanced congruence. As a result the both retarded and advanced fields should be taken into account for the domain wall boundary conditions at  $r = r_{reg}$ .

However, the antipodal relation  $Y^+ = -1/\bar{Y}^-$  between the *retarded* KN solution on the positive physical sheet  $Y^+(x, \tau^-)$  and the *advanced* conjugate solution  $\bar{Y}^-$  on the negative sheet (see [42, ?]) allows us to consider these solutions as analytically related, displaying close analogue with formation of the Feynman propagator from a unique analytic multisheeted solution.

Using this analogue, one can define the physical space-time of a spinning particle by analytical connection of two of the four physical sheets of the K3 surface. For example, unification of the positive sheet of the retarded  $F^-$ -solution  $Y^+(x, \tau^-)$  with negative sheet of the advanced source (corresponding to  $F^+$ ) may be considered as a physical space-time of some spinning particle, while the ‘physical’ spacetime for the corresponding *anti-particle* could be identified with alternative choice of two physical sheets: positive sheet solution  $Y^-(x, \tau^+)$  of the advanced generating function  $F^+$ , extended analytically on the negative sheet of the retarded generator  $F^-$ .

## 5. Outlook

One sees that the Kerr-Schild geometry displays striking parallelism with basic structures of superstring theory. In particular, one of the striking results is the presence of inherent Calabi-Yau twofold in the complex twistorial structure of the Kerr geometry. In the recent paper [13] we argued that it is not accidental, because gravity is a fundamental part of the superstring theory, and the Kerr-Schild gravity, being based on the twistor theory, Kerr theorem and the wonderful multisheeted complex Kerr geometry [37, 38, 39], displays the most deep inherent structures of space-time which may lie beyond quantum physics.

In many respects the Kerr-Schild gravity represents a version of Witten’s twistor-string theory [41], which is also four-dimensional, based on twistors and related with experimental particle physics. It has been observed recently [42, ?] that the complex Kerr string has much in common with the N=2 critical superstring [22, 21, 47], which is also related with twistors. The N=2 string is complex and has the complex critical dimension two, which corresponds to four real dimensions, indicating that it may lead to four-dimensional theory of superstrings. However,

signature of the  $N=2$  string may only be  $(2,2)$  or  $(4,0)$ , which caused obstacles for embedding of this string in the space-times with Minkowskian signature. Up to our knowledge, the trouble was not resolved, and the initially enormous interest to  $N=2$  string seems to be dampened. Meanwhile, embedding of the  $N=2$  string in the complexified Kerr geometry is very natural and hints that stringlike structures of the real and complex Kerr geometry may be related with underlying dynamics of the  $N=2$  superstring [20, 42].

Along with wonderful parallelism with the standard superstring theory, the stringy system of the four-dimensional KN geometry displays very essential peculiarities. First of all note that the discussed regular source of the KN solution provides consistent matching with quantum theory, since it leads to a flat space-time in the Compton region and predicts the Compton size of the dressed electron and some other of its remarkable features [12]. Among them is the discussed above closed Kerr string positioned on the boundary of the Compton region. This string is lightlike and similar to Discrete Light Cone Quantization (DLCQ) circle of M(atric) theory [46]. It has been noted in [48] that such a string cannot be closed indeed. This circular string should have the end points (D0-branes) which may be in the same space position, but have to be separated by a time-interval. It is supported by the discussed in Sec. 2 singular pole which appears as a consequence of the wave excitations of the gravitating soliton model. Therefore, along with the stringy D1-brane of the KN source, there appears the adjoined lightlike D0-brane (which is indeed a pair of D0-brane-antibrane system), and the extended source takes the form of a complex of the D0-D1-D2-D3 branes. The singular lightlike D0-component describes zitterbewegung of the point-like electron, while the complex of D1-D2-D3-branes determines its extended dressed shape [13].

The considered stringy structures of the real and complex Kerr geometry set a parallelism between the 4d Kerr geometry and superstring theory, indicating the potential role of this alliance in the particle physics. On the other hand, these relationships show that complexification of the Kerr geometry may serve as an alternative to traditional compactification of higher dimensions in superstring theory.

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